Self-organization and phase transition in traffic-flow model of a two-lane roadway



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LETTER TO THE EDITOR

# Self-organization and phase transition in traffic-flow model of a two-lane roadway

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**Abstract.** A deterministic cellular automaton model is presented to simulate the traffic flow in a two-lane roadway. The model is an extended version of the one-dimensional asymmetric exclusion model to take into account the exchange of cars between the first and the second lanes. Using computer simulation, it is shown that the exchange of cars has an important effect on phase transition between the maximal velocity phase and the high-density phase. The phase diagram of the phase transition is found. Also, a simple mean-field theory is presented to analyse the traffic flow of a two-lane roadway.

Recently, traffic problems have attracted considerable attention. Traffic simulations based on various hydrodynamic models have provided much insight [1, 2]. However, the simulation of traffic flow in an entire city is a formidable task since it involves many degrees of freedom. Cellular automaton (CA) models are being used increasingly in simulations of complex physical systems [3, 4]. In some complex systems, the cellular automaton models provide only some general qualitative features of the system while in other cases useful quantitative information can be obtained.

Very recently, Biham *et al* [5] presented a simple model which describes traffic flow in two dimensions. They found that a dynamical jamming transition occurs at the critical density $p_c \approx 0.3$–$0.4$ of cars. The jamming transition separates the low-density moving phase in which all cars move and the high-density jamming phase in which all cars are stopped. The CA model proposed by Biham *et al* [5] is a two-dimensional version of the one-dimensional asymmetric exclusion model. Recently, the 1D asymmetric exclusion model has been extensively studied to give an understanding of systems of interacting particles [6–8]. The 1D exclusion model is used to study the microscopic structure of shocks [9, 10], is closely linked to growth processes [11–13] and can also be formulated as traffic jam or queuing problems [14]. The 1D asymmetric exclusion model is one of the simplest examples of a driven diffusive system [15, 16].

In this letter, we consider traffic flow in a two-lane roadway. We extend the 1D asymmetric exclusion model to take into account the exchanges of cars between the first and the second lanes. We study the effect of interaction between the traffic flow in the first lane and that in the second lane on the phase transition. In our traffic-flow model of a two-lane roadway, cars moving on the first (second) lane can shift to the second (first) lane if a car is blocked by another car. In the limit of no interaction between the first and second lanes, our model for traffic flow in a two-lane roadway reproduces the 1D asymmetric exclusion model. We consider the extension of the 1D asymmetric simple-exclusion process in which cars jump at unit rate with probability 1 to a vacant neighbouring site on the up. The deterministic CA model with a periodic boundary of the 1D

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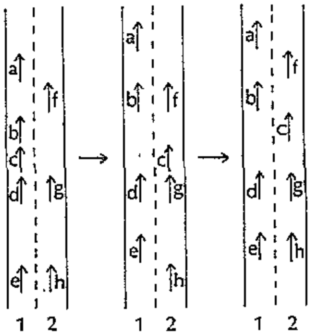


**Figure 1.** The cellular automaton model for traffic flow in a two-lane roadway. The procedure is shown to move cars in two time steps. On an odd time step, the cars in the first lane move ahead, shift to the second lane or stop according to CA rules. Then, on an even time step, the cars in the second lane move ahead, shift to the first lane or stop according to the CA rules.

asymmetric simple-exclusion models is consistent with Wolfram's CA rule no. 184 [3, 5]. For later convenience, we describe the 1D model and present the result [5]. In one dimension there are cars moving along a closed ring. On every time step, each car moves to the up unless it is blocked by another car. The asymptotic mean velocity $\langle v \rangle$ is independent of initial conditions. It is $\langle v \rangle = 1$ for the density $p < \frac{1}{2}$, while for $p > \frac{1}{2}$ it decreases continuously to zero according to $\langle v \rangle = (1-p)/p$. There is a phase transition between the maximal velocity phase in which $\langle v \rangle = 1$ and the high-density phase in which $\langle v \rangle = (1-p)/p$.

We extend the 1D model to the model describing traffic flow in a two-lane roadway. Our CA model is defined on two one-dimensional lattices of $2 \times n$ sites with periodic boundary conditions. The traffic-flow model is given by a two-state CA model on the two one-dimensional lattices which represent a two-lane roadway (figure 1). Each site contains either an arrow pointing to the up or empty. The arrow pointing to the up represents the car moving to the up. On odd time steps, in the first lane, each arrow moves one step to the up unless the up nearest-neighbouring site is occupied by another arrow. If an arrow is blocked ahead by another arrow on the first lane and its right nearest neighbour on the second lane is occupied, it does not move even if the blocking arrow moves out of the site during the same time step. If an arrow is blocked ahead by another arrow on the first lane and its right nearest neighbour on the second lane is unoccupied, it shifts to the second lane. On even time steps, in the second lane, each arrow moves one step to the up unless the up nearest-neighbouring site is occupied by another arrow. If an arrow is blocked ahead by another arrow on the second lane and its left nearest neighbour on the first lane is occupied, it does not move even if the blocking arrow moves out of the site during the same time step. If an arrow is blocked ahead by another arrow on the second lane and its left nearest neighbour on the first lane is unoccupied, it shifts to the first lane. For an illustration, figure 1 indicates the procedure moving cars over two time steps. In this model, the traffic problem is reduced to its simplest form. The essential features are maintained. These features include the simultaneous flow in two parallel directions of cars which cannot overlap and can shift between the first and second lanes. In our models, the total number of cars on a two-lane roadway is conserved. However, the total number of cars in each lane (column) is

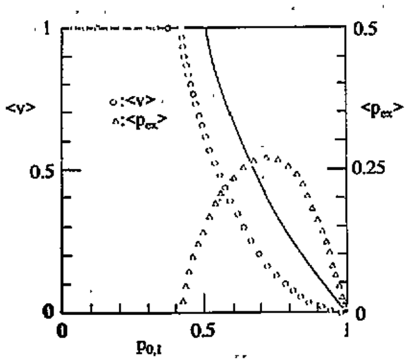


**Figure 2.** The plots of mean velocity $\langle v\rangle$ (indicated by circles) and density $\langle p_{ex}\rangle$ (indicated by triangles) of the exchanged cars against the density $p_{0,1}$ in the case $p_{0,1}=p_{0,2}$. For comparison, the velocity of the one-dimensional model is indicated by the solid curve.

not conserved since the cars can shift to another lane when they are blocked ahead by other cars.

We consider the simulation procedure for the CA model explained above. The initial densities of cars on the first and second lanes are given respectively by $p_{0,1}$ and $p_{0,2}=p_{0,1}f$ where $f$ is the fraction of cars on the second lane. Initially, cars are randomly distributed at the sites on the two-lane roadway with densities $p_{0,1}$ and $p_{0,2}$. The cars on the first lane move ahead, shift to the second lane or stop on odd time steps according to the CA rules explained above. Then, on even time steps, the cars in the second lane move ahead, shift to the first lane or stop according to the CA rules. We have performed simulations of the CA model starting with an ensemble of random initial conditions where the system size is $2\times 1000$, $p_{0,1}=0.0$–$1.0$ and $f=0.0$–$1.0$. Each run is calculated for 4000 time steps.

We present the simulation result obtained by the procedure explained above. First, we consider the traffic flow with equal densities $p_{0,1}=p_{0,2}$ ($f=1.0$). Figure 2 shows the plot of mean velocity $\langle v\rangle$ of cars against the density $p_{0,1}$ of cars (mean velocity $\langle v\rangle$ plotted by circles). The mean velocity $\langle v\rangle$ of cars in the two-unit time interval is defined to be the number of cars moving successfully in the time interval divided by the total number of cars. The traffic flow shows a periodic motion after a number of time steps. The mean velocity $\langle v\rangle$ is calculated after the traffic flow reaches the periodic motion. The velocity $\langle v\rangle$ has a maximum value $\langle v\rangle=1$, indicating that no cars are ever blocked, while $\langle v\rangle=0$ means that all the cars are stopped and never move at all. The velocity of cars maintains maximal velocity ($\langle v\rangle=1$) until the density $p_{0,1}$ reaches the transition point $p_c=0.42\pm0.01$. All the cars in each lane move ahead without blocking. The velocity at the limit of no interaction between the lanes is shown by the solid curve in figure 2 where $\langle v\rangle=1$ for $p\leqslant 0.5$ and $\langle v\rangle=(1-p)/p$ for $p>0.5$. With interaction between the first and second lanes, the phase transition between the maximal velocity phase ($\langle v\rangle=1$) and the high-density phase ($\langle v\rangle<1$) occurs at a lower density $p_c=0.42$ and the velocity $\langle v\rangle$ of cars becomes slower than with no interaction. The density $\langle p_{ex}\rangle$ of cars which exchange between the first and second lanes in a unit time interval is plotted as triangles in figure 2. Just above the transition point $p_c=0.42$, the exchange of cars between lanes begins. The number of exchanged cars increases with the density $p_{0,1}$, reaches a maximum value at $p_{0,1}\approx 0.72$ and then decreases with $p_{0,1}$. We find that the exchange of cars between lanes has an important effect on the phase transition.

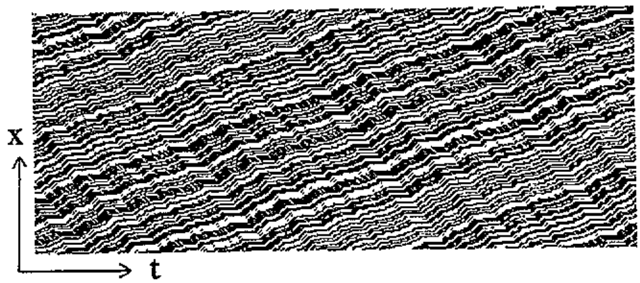


**Figure 3** The space-time pattern ($x=200$, $t=1,3,5,\ldots,999$) for $p_{0,1}=p_{0,2}=0.6$. After odd time steps, the cars in the first lane are plotted by the dot.

Figure 3 shows the space-time pattern ($x=200$ and $t=1,3,5,\ldots,999$) for $p_{0,1}=p_{0,2}=0.6$ where $\langle v\rangle=0.3466$. The cars in the first lane after odd time steps are plotted as dots. The pattern shows a modulated periodic behaviour. Its behaviour is due to the periodic exchange of cars between the first and second lanes. The space-time pattern is characteristic of the traffic flow in a two-lane roadway.

Second, we consider the traffic flow with no cars in the second lane ($p_{0,2}=0$). Figure 4 shows the plot of the mean velocity $\langle v\rangle$ of cars against the density $p_{0,1}$ of cars (mean velocity $\langle v\rangle$ is plotted as circles). The velocity of cars maintains maximum velocity ($\langle v\rangle=1$) until the density $p_{0,1}$ reaches the transition point $p_c=0.67\pm0.01$. The phase transition between the maximum velocity phase ($\langle v\rangle=1$) and the high-density phase ($\langle v\rangle<1$) occurs at higher density $p_c=0.67$ than for the 1D model. The velocity $\langle v\rangle$ of cars becomes faster than that of no second lane. The mean densities $p_{1,\mathrm{I}}$ and $p_{1,\mathrm{O}}$ on the first lane before and after the exchange of cars between the first and second lanes are plotted, respectively, by triangles and squares in figure 4. Below the transition point $p_c=0.67$, the densities of cars in the first and second lanes become equivalent after several time steps. Above the transition point, the mean density $p_1$ in the first lane changes periodically on each time step. The number of exchanged cars increases with the initial density $p_{0,1}$. As the initial density $p_{0,1}$ approaches 1, most cars shift alternately to another lane. In the limit of $p_{0,1}=1$, all cars shift alternately to another lane. In this

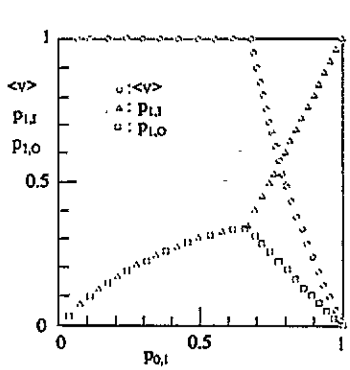


**Figure 4** The plots of the mean velocity $\langle v\rangle$ (indicated by circles) against the density $p_{0,1}$ for $p_{0,2}=0$. The mean densities $p_{1,\mathrm{I}}$ and $p_{1,\mathrm{O}}$ in the first lane before and after the exchanges of cars between the first and second lanes are plotted, respectively, by triangles and squares.

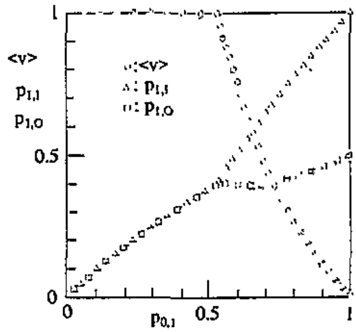


**Figure 5.** The plots of the mean velocity $\langle v \rangle$ (circles), the mean densities $p_{1,\mathrm{I}}$ (triangles) and $p_{1,\mathrm{O}}$ (squares) against the density $p_{0,1}$ for $p_{0,2}=0.5p_{0,1}$.

limit, the cars do not move but vibrate left and right. Figure 5 shows the plots of mean velocity $\langle v \rangle$, the mean densities $p_{1,\mathrm{I}}$ and $p_{1,\mathrm{O}}$ before and after the exchange of cars against the initial density $p_{0,1}$ for $p_{0,2}=p_{0,1}/2(f=0.5)$. The phase transition between the maximal velocity phase and the high-density phase occurs at the transition point $p_c=0.52\pm0.01$. The number of exchanged cars, which is proportional to the difference $p_{1,\mathrm{I}}-p_{1,\mathrm{O}}$, is maintained at zero until the initial density $p_{0,1}$ reaches the transition $p_c=0.52$. Above the transition point $p_c=0.52$, the number of exchanged cars increases with the initial density $p_{0,1}$.

We show the phase diagram of the traffic flow in the two-lane roadway in figure 6, which indicates the relationship between the initial density $p_{0,1}$ in the first lane and the fraction $f$ in the second lane ($p_{0,2}=p_{0,1}f$) (the transition points are plotted as full circles). The region on the left-hand side of the transition line represents the maximum velocity phase ($\langle v \rangle=1$). The region on the right-hand side of the transition line represents the high-density phase ($\langle v \rangle<1$). The transition point increases with decreasing $f$. We find that the traffic flow on the two-lane roadway shows an interesting behaviour.

We present a simple mean-field approach to the phase transition. We derive the mean velocity $\langle v \rangle$ and the density $\langle p_{\mathrm{ex}} \rangle$ of exchanged cars. In the one-dimensional case, the mean velocity $\langle v \rangle$ is known to be given by $\langle v \rangle=1$ for $p\leqslant0.5$ and $\langle v \rangle=$

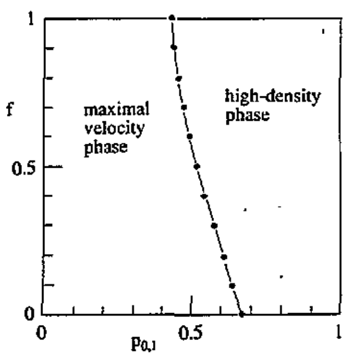


**Figure 6.** The phase diagram for the phase transition between the maximum velocity phase ($\langle v \rangle=1$) and the high-density phase ($\langle v \rangle<1$). The region on the left-hand side of the transition line represents the maximum velocity phase. The region on the right-hand side of the transition line represents the high-density phase.

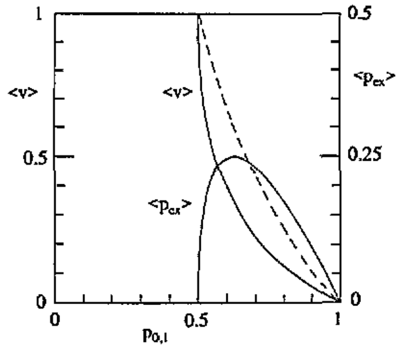


**Figure 7.** The mean velocity $\langle v \rangle$ and the density $\langle p_{ex} \rangle$ of exchanged cars obtained by the mean-field theory. For comparison, the velocity in the one-dimensional model is indicated by the dotted line.

$(1-p)/p$ for $p>0.5$ [5]. In our model, the number of cars on each lane changes periodically. In the case $p_{0,1}=p_{0,2}$, the maximum value $p_{max}$ of the density in the first (or second) lane is approximately given by

$$p_{max}=p_{0,1}+\langle p_{ex}\rangle/2. \tag{1}$$

The mean velocity $\langle v \rangle$ is determined by the maximal density $p_{max}$ since the density on the first lane on an odd time step equals to $p_{max}$. By replacing the density in $\langle v \rangle = (1-p)/p$ with $p_{max}$, the mean velocity $\langle v \rangle$ is given by

$$\begin{aligned} &\langle v \rangle = 1 \qquad \text{for } p_{max} \leqslant 0.5 \\ &\langle v \rangle = (1-p_{max})/p_{max} \qquad \text{for } p_{max}>0.5. \end{aligned} \tag{2}$$

We calculate the mean density $\langle p_{ex} \rangle$ of exchanged cars self-consistently. The density of stopped cars is given by $1-\langle v \rangle$. Then, the mean density $\langle p_{ex} \rangle$ of cars which can shift into another lane is given by the product of the density of stopped cars and the density of unoccupied sites:

$$\langle p_{ex}\rangle=(1-p_{0,1})(1-\langle v \rangle). \tag{3}$$

By solving equation (3), the density $\langle p_{ex} \rangle$ of exchanged cars is obtained

$$\begin{aligned} &\langle p_{ex}\rangle = 0 \qquad \text{for } p_{0,1} \leqslant 0.5 \\ &\langle p_{ex}\rangle = (1-2p_{0,1})+(2p_{0,1}-1)^{1/2} \qquad \text{for } p_{0,1}>0.5. \end{aligned} \tag{4}$$

The mean velocity is obtained as

$$\begin{aligned} &\langle v \rangle = 1 \qquad \text{for } p_{0,1} \leqslant 0.5 \\ &\langle v \rangle = (1-p_{0,1}-\langle p_{ex}\rangle/2)/(p_{0,1}+\langle p_{ex}\rangle/2) \qquad \text{for } p_{0,1}>0.5. \end{aligned} \tag{5}$$

Figure 7 shows the plot of the mean velocity $\langle v \rangle$ and the density $\langle p_{ex} \rangle$ of exchanged cars against the density $p_{0,1}$. For comparison, the velocity $\langle v \rangle = (1-p_{0,1})/p_{0,1}$ in the one-dimensional model is also indicated by the dotted line. The velocity distribution obtained from the mean-field theory is compared with figure 2. The mean-field theory can explain qualitatively the mean velocity and the density of exchanged cars. However, quantitatively, the result of the mean-field theory is not consistent with the simulation result. This inconsistency is due to neglecting the space-time correlation between cars.

In summary, we present a deterministic cellular automaton model for the traffic flow in a two-lane roadway, and show that the interaction of traffic flow between the first and second lanes has an important effect on the phase transition. We give a simple mean-field theory to analyse the traffic flow of a two-lane roadway.

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